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Approximate solutions of stochastic differential delay equations with Markovian switching

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Abstract

Recently, stochastic differential equations with Markovian switching (SDEwMS) have received a great deal of attention. In this paper, the Euler–Maruyama method is developed, one of the most powerful numerical schemes, for the stochastic differential delay equations with Markovian switching (SDDEwMS).

Keywords: Brownian motion; Euler–Maruyama method; Lipschitz condition; Markov chain generator

1. Introduction

Throughout this paper, we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets). Let $w(t) = (w_t^1, \ldots, w_t^d)^{\mathrm{T}}$, $t\geqslant 0$, be a d-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean vector norm as well as the matrix trace norm. Let $\tau > 0$ and $C([-\tau, 0]; \mathfrak{R}^m)$ denote the family of continuous function φ from $[-\tau, 0]$ to $\mathfrak{R}^m$ with the norm $\|\varphi\| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$. Denote by $C^b_{\mathscr{F}_0}([-\tau, 0]; \mathfrak{R}^m)$ the family of all bounded, $\mathscr{F}_0$-measurable, $C([-\tau, 0]; \mathfrak{R}^m)$-valued random variables. If $X(t)$ is a continuous $\mathfrak{R}^m$-valued stochastic process on $t \in [-\tau, \infty)$, we let $X_t = \{X(t+\theta) : -\tau\leqslant\theta\leqslant 0\}$ for all $t\geqslant 0$, which is regarded as a $C([-\tau, 0]; \mathfrak{R}^m)$-valued stochastic process. Let $p > 0$, and denote by $L^p_{\mathscr{F}_0}([-\tau, 0]; \mathfrak{R}^m)$ the family of $\mathscr{F}_0$-measurable

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$C([-\tau,0];\Re^m)$-valued random variables such that $E\|\xi\|^p<\infty$. Let $r(t),\ t\geqslant 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S=\{1,2,\ldots,N\}$ with generator $\Gamma=(\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta)=j|r(t)=i\}=\begin{cases}\gamma_{ij}\delta+\mathrm{o}(\delta) & : \quad \text{if } i\neq j,\\ 1+\gamma_{ii}\delta+\mathrm{o}(\delta) & : \quad \text{if } i=j,\end{cases}$$

where $\delta>0$. Here $\gamma_{ij}\geqslant 0$ is transition rate from i to j if $i\neq j$ while

$$\gamma_{ii}=-\sum_{j\neq i}\gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is right continuous step function with a finite number of sample jumps in any finite subinterval of $\Re_+$.

We consider the following stochastic differential delay equation with Markovian switching (SDDEwMS):

$$\mathrm{d}y(t)=f(y(t),y(t-\tau),r(t))\,\mathrm{d}t+g(y(t),y(t-\tau),r(t))\,\mathrm{d}w(t),\quad 0\leqslant t\leqslant T, \tag{1.1}$$

with initial data $y(0)=\xi\in C([-\tau,0],\Re^m)$ and $r(0)=i_0\in S$, where $f:\Re^m\times\Re^m\times S\to\Re^m$ and $g:\Re^m\times\Re^m\times S\to\Re^{m\times d}$. We assume that (1.1) has a unique solution. We refer the reader to Mao [7] and Skorohod [13] for the conditions on the existence and uniqueness of the solution. For the background of SDDEwMS, we refer to, for example, Ji and Chizeck [4], Mao [7,10], Mariton [11], Shaikhet [12], and Skorohod [13], a few to name.

The numerical methods on stochastic differential equations (SDE) and stochastic functional differential equations (SFDE) have also been well established and we refer the reader to, for example, Hu [3], Higham et al. [2], Klodeden and Platen [5], Mao [6,8] and the references therein. Our aim is to develop the Euler–Maruyama method for Eq. (1.1).

To define the Euler–Maruyama approximate solution, we will need the property of embedded discrete Markov chain. The following lemma describes this property.

Lemma 1.1. *Let $r_k^\Delta=r(k\Delta)$ for $\Delta>0$ and $k\geqslant 0$. Then $\{r_k^\Delta,\ k=0,1,2,\ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta)=(P_{ij}(\Delta))_{N\times N}=\mathrm{e}^{\Delta\Gamma}.$$

Proof. Obviously $\{r_k^\Delta,\ k=0,1,2,\ldots\}$ is a Markov chain. Let $P_{ij}(t)$ denote the transition probability of $r(t)$, that is

$$P_{ij}(t)=P\{r(t+s)=j|r(s)=i\},\quad 0\leqslant s\leqslant t<\infty.$$

Proposition 2.10 of Anderson [1] shows that

$$P_{ij}(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\gamma_{ij}^{(n)}, \tag{1.2}$$

where $\gamma_{ij}^{(n)}$ denotes the i, jth component of the nth power Γ^n of Γ. Hence

$$P_{ij}(\Delta) = P\{r((k+1)\Delta) = j | r(k\Delta) = i\} = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} \gamma_{ij}^{(n)} = (e^{\Gamma\Delta})_{ij},$$

as required. □

Given a stepsize $\Delta > 0$, the discrete Markov chain $\{r_k^\Delta,\ k = 0, 1, 2, \ldots\}$ can be simulated as follows: Compute the one-step transition probability matrix

$$P(\Delta) = (P_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}.$$

Let $r_0^\Delta = i_0$ and generate a random number ζ_1 which is uniformly distributed in [0, 1]. If $\zeta_1 = 1$ then let $r_1^\Delta = i_1 = N$ or otherwise find the unique integer $i_1 \in S$ for

$$\sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leqslant \zeta_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta)$$

and let $r_1^\Delta = i_1$, where we set $\sum_{i=1}^{0} P_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ζ_2 which is again uniformly distributed in [0, 1]. If $\zeta_2 = 1$ then let $r_2^\Delta = i_2 = N$ or otherwise find the unique integer $i_2 \in S$ for

$$\sum_{j=1}^{i_2-1} P_{i_1,j}(\Delta) \leqslant \zeta_2 < \sum_{j=1}^{i_2} P_{i_1,j}(\Delta)$$

and let $r_2^\Delta = i_2$. Repeating this procedure a trajectory of $\{r_k^\Delta,\ k = 0, 1, 2, \ldots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories.

After explaining how to simulate the discrete Markov chain $\{r_k^\Delta, k = 1, 2, \ldots\}$, we can now define the Euler–Maruyama (EM) approximate solution to the SDDEwMS (1.1). Given a stepsize $0 < \Delta < \tau$, let $t_k = k\Delta$. Let $[a/\Delta]$ denotes the integer part of the real number a/Δ ($a \in \Re$). Compute the discrete approximations $X(t) \approx y(t_k)$ by setting $X_0(t) = \xi(t)$ on $-\tau \leqslant t \leqslant 0$, $r_0^\Delta = i_0$ and forming (more details see [8])

$$X_{k+1} = X_k + f(X_k, X_{[(k\Delta-\tau)/\Delta]}, r_k^\Delta)\Delta + g(X_k, X_{[(k\Delta-\tau)/\Delta]}, r_k^\Delta)\Delta w_k, \tag{1.3}$$

where $\Delta w_k = w(t_{k+1}) - w(t_k)$. Let

$$\bar{X}_1(t) = X_k, \quad \bar{X}_2(t) = X_{[(k\Delta-\tau)/\Delta]}, \quad \bar{r}(t) = r_k^\Delta \qquad \text{for } t \in [t_k, t_{k+1}) \tag{1.4}$$

and define the continuous EM approximate solution

$$X(t) = X_0 + \int_0^t f(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))\,ds + \int_0^t g(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))\,dw(s). \tag{1.5}$$

Note that $X(t_k) = X_k$, for every $k = 0, 1, \ldots, [T/\Delta]$.

Let us now present a lemma for future use.

Lemma 1.2. *Assume that initial data* $\xi \in L^p_{\mathscr{F}_0}([-\tau, 0]; \Re^m)$ *and*, f *and* g *satisfy the linear growth condition*:

(LG) *There is a constant* $h > 0$ *such that*

$$|f(x, y, i)| \vee |g(x, y, i)| \leqslant h(1 + |x| + |y|) \quad \forall (x, y, i) \in \Re^m \times \Re^m \times S.$$

Then for any $p \geqslant 2$ *there is a constant* K, *which is dependent on only* p, T, h, ξ *but independent of* Δ, *such that the exact solution and the EM approximate solution to the SDDEwMS* (1.1) *have the property that*

$$E\left[\sup_{0\leqslant t\leqslant T} |y(t)|^p\right] \vee E\left[\sup_{0\leqslant t\leqslant T} |X(t)|^p\right] \leqslant K. \tag{1.6}$$

We omit the proof because it is similar to that for stochastic delay differential equations.

2. Convergence with the global Lipschitz condition

In this section we shall show the strong convergence of the EM approximate solution to the exact solution under the following global Lipschitz condition:

(GL) There is a constant $L > 0$ such that

$$|f(x, y, i) - f(\bar{x}, \bar{y}, i)| \vee |g(x, y, i) - g(\bar{x}, \bar{y}, i)| \leqslant L(|x - \bar{x}| + |y - \bar{y}|)$$

for all $x, y, \bar{x}, \bar{y} \in \Re^m$ and $i \in S$.

Note from this global Lipschitz condition that

$$|f(x, y, i)| \leqslant |f(x, y, i) - f(0, 0, i)| + |f(0, 0, i)| \leqslant L(|x| + |y|) + |f(0, 0, i)|$$

and

$$|g(x, y, i)| \leqslant |g(x, y, i) - g(0, 0, i)| + |g(0, 0, i)| \leqslant L(|x| + |y|) + |g(0, 0, i)|.$$

Hence

$$|f(x, y, i)| \vee |g(x, y, i)| \leqslant h(1 + |x| + |y|) \tag{2.1}$$

with $h = L \vee \max\{|f(0, 0, i)| \vee |g(0, 0, i)| : i \in S\}$. In other words, the global Lipschitz condition (GL) implies the linear growth condition (LG). Lemma 1.2 then shows that under condition (GL) any pth moments, especially the 2nd moments, of the exact solution and the EM approximate solution to SDDEwMS (1.1) are finite.

Theorem 2.1. *Under the global Lipschitz condition* (*GL*),

$$E\left[\sup_{0\leqslant t\leqslant T} |X(t) - y(t)|^2\right] \leqslant C\Delta + \mathrm{o}(\Delta), \tag{2.2}$$

where C is a positive constant independent of Δ.

Proof. From (GL) we obtain

$$
\begin{aligned}
&E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(y(s),y(s-\tau),s,r(s))|^2\,\mathrm{d}s\\
&\leqslant 2E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),r(s)) - f(y(s),y(s-\tau),s,r(s))|^2\,\mathrm{d}s\\
&\quad + 2E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s\\
&\leqslant 4L^2\int_0^t E\left(|\bar{X}_1(s)-y(s)|^2 + |\bar{X}_2(s)-y(s-\tau)|^2\right)\mathrm{d}s\\
&\quad + 2E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s. \qquad (2.3)
\end{aligned}
$$

Let $n=[T/\Delta]$, the integer part of T/Δ. Then

$$
\begin{aligned}
&E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s\\
&=\sum_{k=0}^{n} E\int_{t_k}^{t_{k+1}} |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s \qquad (2.4)
\end{aligned}
$$

with t_{n+1} being now set to be T. Let I_G be the indicator function for set G. Moreover, in the remainder of the proof C is a positive constant independent of Δ which may change line by line. With these notations we derive, using (GL), that

$$
\begin{aligned}
&E\int_{t_k}^{t_{k+1}} |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s\\
&\leqslant 2E\int_{t_k}^{t_{k+1}} |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s)) - f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2 I_{\{r(s)\neq r(t_k)\}}\,\mathrm{d}s\\
&\leqslant CE\int_{t_k}^{t_{k+1}} [1+|\bar{X}_1(s)|^2+|\bar{X}_2(s)|^2] I_{\{r(s)\neq r(t_k)\}}\,\mathrm{d}s\\
&\leqslant C\int_{t_k}^{t_{k+1}} E[E[(1+|\bar{X}_1(s)|^2+|\bar{X}_2(s)|^2) I_{\{r(s)\neq r(t_k)\}}|r(t_k)]]\,\mathrm{d}s\\
&= C\int_{t_k}^{t_{k+1}} E[E[(1+|\bar{X}_1(s)|^2+|\bar{X}_2(s)|^2)|r(t_k)]E[I_{\{r(s)\neq r(t_k)\}}|r(t_k)]]\,\mathrm{d}s,
\end{aligned}
$$

where in the last step we use the fact that $\bar{X}_1(s)=X_1(t_k)$, $\bar{X}_2(s)=X([t_k-\Delta/\Delta]\Delta)$ and $I_{\{r(s)\neq r(t_k)\}}$ when $t_k\leqslant s\leqslant t_{k+1}$ are conditionally independent with respect to the σ-algebra generated by $r(t_k)$. But, by the Markov property,

$$
\begin{aligned}
E[I_{\{r(s)\neq r(t_k)\}}|r(t_k)] &= \sum_{i\in S} I_{\{r(t_k)=i\}} P(r(s)\neq i|r(t_k)=i) \\
&= \sum_{i\in S} I_{\{r(t_k)=i\}} \sum_{j\neq i}(\gamma_{ij}(s-t_k)+\mathrm{o}(s-t_k)) \\
&\leqslant \left(\max_{1\leqslant i\leqslant N}(-\gamma_{ii})\varDelta+\mathrm{o}(\varDelta)\right)\sum_{i\in S} I_{\{r(t_k)=i\}} \leqslant C\varDelta+\mathrm{o}(\varDelta).
\end{aligned}
\tag{2.5}
$$

So, by Lemma 1.2,

$$
\begin{aligned}
&E\int_{t_k}^{t_{k+1}} |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s \\
&\quad\leqslant (C\varDelta+\mathrm{o}(\varDelta))\int_{t_k}^{t_{k+1}}[1+E|\bar{X}_1(s)|^2+E|\bar{X}_2(s)|^2]\,\mathrm{d}s\leqslant \varDelta(C\varDelta+\mathrm{o}(\varDelta)).
\end{aligned}
$$

Substituting this into (2.4) gives

$$
E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-f(\bar{X}_1(s),\bar{X}_2(s),r(s))|^2\,\mathrm{d}s\leqslant C\varDelta+\mathrm{o}(\varDelta). \tag{2.6}
$$

Combining (2.3) and (2.6) we obtain that

$$
\begin{aligned}
&E\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-f(y(s),y(s-\tau),s,r(s))|^2\,\mathrm{d}s \\
&\quad\leqslant 4L^2\int_0^t E\left(|\bar{X}_1(s)-y(s)|^2+|\bar{X}_2(s)-y(s-\tau)|^2\right)\,\mathrm{d}s+C\varDelta+\mathrm{o}(\varDelta).
\end{aligned}
\tag{2.7}
$$

Similarly, we can show that

$$
\begin{aligned}
&E\int_0^t |g(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-g(y(s),y(s-\tau),s,r(s))|^2\,\mathrm{d}s \\
&\quad\leqslant 4L^2\int_0^t E\left(|\bar{X}_1(s)-y(s)|^2+|\bar{X}_2(s)-y(s-\tau)|^2\right)\,\mathrm{d}s+C\varDelta+\mathrm{o}(\varDelta).
\end{aligned}
\tag{2.8}
$$

Using (2.7), (2.8), the Hölder inequality and the Doob inequality, it is not difficult to show that for $0 \leqslant t \leqslant T$,

$$\begin{aligned}
&E\left(\sup_{0\leqslant s\leqslant t}|X(s)-y(s)|^2\right)\\
&\quad\leqslant 2E\left|\int_0^t f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-f(y(s),y(s-\tau),r(s))\,\mathrm{d}s\right|^2\\
&\qquad+2E\left|\int_0^t g(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-g(y(s),y(s-\tau),r(s))\,\mathrm{d}B(s)\right|^2\\
&\quad\leqslant 2TE\int_0^t |f(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-f(y(s),y(s-\tau),r(s))|^2\,\mathrm{d}s\\
&\qquad+8E\int_0^t |g(\bar{X}_1(s),\bar{X}_2(s),\bar{r}(s))-g(y(s),y(s-\tau),r(s))|^2\,\mathrm{d}s\\
&\quad\leqslant C\int_0^t E\left(\sup_{0\leqslant r\leqslant s}|X(r)-y(r)|^2\right)+C\int_0^T E|\bar{y}(s)-y(s-\tau)|^2\,\mathrm{d}s+C\Delta+\mathrm{o}(\Delta),
\end{aligned}\tag{2.9}$$

where

$$\bar{y}(t)=y([(k\Delta-\tau)/\Delta]\Delta).$$

By Mao [6, p. 164, Lemma 5.2], we know that

$$\int_0^T E|\bar{y}(s)-y(s-\tau)|^2\,\mathrm{d}s\leqslant C\Delta+\mathrm{o}(\Delta).\tag{2.10}$$

Putting (2.10) into (2.9) we see that

$$E\left(\sup_{0\leqslant s\leqslant t}|X(s)-y(s)|^2\right)\leqslant C\int_0^t E\left(\sup_{0\leqslant r\leqslant s}|X(r)-y(r)|^2\right)+C\Delta+\mathrm{o}(\Delta),$$

and the required result (2.2) follows from the Gronwall inequality. □

3. Convergence with the local Lipschitz and linear growth condition

In this section we shall discuss the strong convergence of the EM method on the SDDEwMS (1.1) under the local Lipschitz condition. In many situations, the coefficients f and g are only locally Lipschitz continuous. It is therefore useful to establish the strong convergence of the EM method under the local Lipschitz condition. By the local Lipschitz condition we mean:

(LL) For each $R=1,2,\ldots$, there is a constant $L_R>0$ such that

$$|f(x,y,i)-f(\bar{x},\bar{y},i)|\vee|g(x,y,i)-g(\bar{x},\bar{y},i)|\leqslant L_R(|x-\bar{x}|+|y-\bar{y}|)$$

for all $i\in S$ and those $x,y,\bar{x},\bar{y}\in\Re^m$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\leqslant R$.

Theorem 3.1. *Under the local Lipschitz condition* (*LL*) *and the linear growth condition* (*LG*), *the EM approximate solution converges to the exact solution of the SDDEwMS* (1.1) *in the sense that*

$$\lim_{\Delta\to 0} E\left[\sup_{0\leqslant t\leqslant T} |X(t)-y(t)|^2\right]=0. \tag{3.1}$$

Proof. The techniques of the proof have been developed in Higham et al. [2] where they showed the strong convergence of the EM method for the SDE *without Markov switching* under the local Lipschitz condition. We therefore only give the outline of the proof here.

Fix a $p>2$. By Lemma 1.2, there is a positive constant K independent of Δ such that

$$E\left[\sup_{0\leqslant t\leqslant T} |X(t)|^p\right] \vee E\left[\sup_{0\leqslant t\leqslant T} |y(t)|^p\right]\leqslant K. \tag{3.2}$$

For sufficiently large integer R, define the stopping times

$$\tau_R=\inf\{t\in[0,T]:|X(t)|\geqslant R\},\quad \rho_R=\inf\{t\in[0,T]:|y(t)|\geqslant R\},\quad \theta_R=\tau_R\wedge\rho_R,$$

where throughout this paper we set $\inf\emptyset=T$. Let

$$e(t)=X(t)-y(t).$$

Recall the Young inequality: for $r^{-1}+q^{-1}=1$ and $\forall a,b,\delta$

$$ab\leqslant\frac{\delta}{r}a^r+\frac{1}{q\delta^{q/r}}b^q.$$

Thus, for any $\delta>0$,

$$\begin{aligned}
E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2\right] &= E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2 I_{\{\tau_R>T,\rho_R>T\}}\right]+E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2 I_{\{\tau_R\leqslant T \text{ or } \rho_R\leqslant T\}}\right]\\
&\leqslant E\left[\sup_{0\leqslant t\leqslant T}|e(t\wedge\theta_R)|^2 I_{\{\theta_R>T\}}\right]+\frac{2\delta}{p}E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^p\right]\\
&\quad+\frac{1-2/p}{\delta^{2/(p-2)}}P(\tau_R\leqslant T \quad\text{or}\quad \rho_R\leqslant T).
\end{aligned}\tag{3.3}$$

Now, by (3.2),

$$P(\tau_R\leqslant T)=E\left[I_{\{\tau_R\leqslant T\}}\frac{|X(\tau_R)|^p}{R^p}\right]\leqslant\frac{1}{R^p}E\left[\sup_{0\leqslant t\leqslant T}|X(t)|^p\right]\leqslant\frac{K}{R^p}.$$

A similar result can be derived for ρ_R, so that

$$P(\tau_R\leqslant T \quad\text{or}\quad \rho_R\leqslant T)\leqslant\frac{2K}{R^p}.$$

Note also from (3.2) that

$$E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^p\right]\leqslant 2^{p-1}\left(E\left[\sup_{0\leqslant t\leqslant T}|X(t)|^p\right]+E\left[\sup_{0\leqslant t\leqslant T}|y(t)|^p\right]\right)\leqslant 2^pK.$$

Using these bounds gives

$$E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2\right]\leqslant E\left[\sup_{0\leqslant t\leqslant T}|X(t\wedge\theta_R)-y(t\wedge\theta_R)|^2\right]+\frac{2^{p+1}\delta K}{p}+\frac{2(p-2)K}{p\delta^{2/(p-2)}R^p}. \quad (3.4)$$

In the similar way as Theorem 2.1 was proved, we can show that

$$E\left[\sup_{0\leqslant t\leqslant T}|X(t\wedge\theta_R)-y(t\wedge\theta_R)|^2\right]\leqslant C_R\Delta+o(\Delta), \quad (3.5)$$

where C_R is a constant independent of Δ. Substituting this into (3.4) gives

$$E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2\right]\leqslant C_R\Delta+o(\Delta)+\frac{2^{p+1}\delta K}{p}+\frac{2(p-2)K}{p\delta^{2/(p-2)}R^p}. \quad (3.6)$$

Now, given any $\varepsilon>0$, we can choose δ so that

$$\frac{2^{p+1}\delta K}{p}<\frac{\varepsilon}{3},$$

then choose R sufficiently large for

$$\frac{2(p-2)K}{p\delta^{2/(p-2)}R^p}<\frac{\varepsilon}{3},$$

and finally choose Δ sufficiently small for

$$C_R\Delta+o(\Delta)<\frac{\varepsilon}{3},$$

so that, in (3.6),

$$E\left[\sup_{0\leqslant t\leqslant T}|e(t)|^2\right]<\varepsilon$$

as required. □

We remark that although Theorem 3.1 does not reveal the order of convergence as Theorem 2.1 did, its proof is optimal in the sense that in the globally Lipschitz case ($L_R \leqslant L$ for all R) we have $C_R = C$ in (3.5) and hence may take $\delta = \Delta$ and $R = \Delta^{-1/(p-2)}$ in (3.6) to recover the result

$$E\left[\sup_{0\leqslant t\leqslant T} |e(t)|^2\right] \leqslant C\Delta + \mathrm{o}(\Delta)$$

obtained in Theorem 2.1.

We observe that the proof of Theorem 3.1 uses only the local Lipschitz condition (LL) and the bounded pth moment property (3.2), namely

(BM) For some $p > 2$, there is a positive constant K independent of Δ such that

$$E\left[\sup_{0\leqslant t\leqslant T} |X(t)|^p\right] \vee E\left[\sup_{0\leqslant t\leqslant T} |y(t)|^p\right] \leqslant K.$$

So the following general statement holds.

Theorem 3.2. *Under the local Lipschitz condition* (*LL*) *and the bounded pth moment condition* (BM), *the EM approximate solution converges to the exact solution of the SDDEwMS* (1.1) *in the sense that*

$$\lim_{\Delta\to 0} E\left[\sup_{0\leqslant t\leqslant T} |X(t) - y(t)|^2\right] = 0.$$

Lemma 1.2 shows that the linear growth condition (LG) implies the bounded pth moment property (BM). It is useful to obtain other alternative conditions that guarantee property (BM) but we shall report them elsewhere.

4. Convergence in probability

Unfortunately the linear growth condition (LG) are often not met by systems of interest. For example, consider the stochastic delay Lotka–Volterra model with Markovian switching

$$\begin{aligned} \mathrm{d}y(t) = \mathrm{diag}(y_1(t), \ldots, y_m(t))([A(r(t))(y(t) - \bar{y}) \\ +\ B(r(t))(y(t-\tau) - \bar{y})]\,\mathrm{d}t + \sigma(r(t))(y(t) - \bar{y})\,\mathrm{d}w(t)), \end{aligned} \tag{4.1}$$

where $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)^{\mathrm{T}} \in \Re^m_+ = \{(x_1, \ldots, x_m)^{\mathrm{T}} \in \Re^m : x_n > 0, n = 1, \ldots, m\}$ is an equilibrium state, $\sigma(i) = (\sigma_{jl}(i))_{m\times m}$ and

$$A(i) = (a_{jl}(i))_{m\times m}, \quad B(i) = (b_{jl}(i))_{m\times m}.$$

This is a special case of the SDDEwMS (1.1), clearly the local Lipschitz condition is satisfied, but the linear growth condition is not satisfied. In this section, we will extend the result of Mao et al. [9] to the SDDEwMS with only the local Lipschitz condition but without the linear growth condition or the bounded pth moment property. The following theorem describes the convergence in probability, instead of L^2, of the EM solutions to the exact solution under some additional conditions in terms of Lyapunov-type functions.

In general, the local Lipschitz condition will only guarantee a unique maximal local solution to Eq. (1.1) for any given initial data ξ and i_0. However, conditions imposed in Theorem 4.1 will guarantee that there is a global solution (see the step (i) of the proof or [14]).

Theorem 4.1. *Let the local Lipschitz condition* (*LL*) *hold. Assume that there exists a* C^2 *function* $V : \Re^m \times S \to \Re_+$ *satisfying the following three conditions*:

(i) *for all* $x \in \Re^m$, $i, j \in S$ *there exists a* $q > 0$, $V(x, i) \leqslant qV(x, j)$;
(ii) $\lim_{|x|\to\infty} V(x, i) = \infty$ *for any* $i \in S$;
(iii) *for some* $\tilde{h} > 0$,

$$\mathscr{L}V(x, y, i) \leqslant \tilde{h}(1 + V(x, i) + V(y, i)) \quad \forall (x, y, i) \in \Re^m \times \Re^m \times S,$$

where

$$\mathscr{L}V(x, y, i) = V_x(x, i) f(x, y, i) + \frac{1}{2}\text{trace}[g^{\mathrm{T}}(x, y, i) V_{xx} g(x, y, i)] + \sum_{j=1}^{N} \gamma_{ij} V(x, j); \tag{4.2}$$

(iv) *for each* $R > 0$ *there exists a positive constant* K_R *such that for all* $i \in S$ *and those* $x, y \in \Re^m$ *with* $|x| \vee |y| \leqslant R$,

$$|V(x, i) - V(y, i)| \vee |V_x(x, i) - V_x(y, i)| \vee |V_{xx}(x, i) - V_{xx}(y, i)| \leqslant K_R |x - y|.$$

Then the EM approximate solution converges to the exact solution of the SDDEwMS (1.1) *in the sense that*

$$\lim_{\Delta\to 0} \left(\sup_{0\leqslant t\leqslant T} |X(t) - y(t)|^2 \right) = 0 \quad \textit{in probability.} \tag{4.3}$$

Proof. We divide the whole proof into three steps.

Step 1: For sufficiently large R, define the stopping time

$$\theta = \inf\{t \in [0, T] : |y(t)| \geqslant R\}.$$

Applying the generalised Itô formula (cf. Mao [7]) to $V(y(t), r(t))$ yields

$$E[V(y(t\wedge\theta), r(t\wedge\theta))] = V(\xi(0), i_0) + E\int_0^{t\wedge\theta} \mathscr{L}V(y(s), y(s-\tau), r(s))\,\mathrm{d}s.$$

By condition (i) and (iii)

$$\begin{aligned} &E[V(y(t\wedge\theta), r(t\wedge\theta))] \\ &\quad\leqslant V(\xi(0), i_0) + \tilde{h}E\int_0^{t\wedge\theta} (1 + V(y(s), r(s)) + V(y(s-\tau), r(s)))\,\mathrm{d}s \\ &\quad\leqslant V(\xi(0), i_0) + \tilde{h}T + \sup_{i\in S, -\tau\leqslant u\leqslant 0} EV(\xi(u), i)\tilde{h}\tau + \tilde{h}(1+q)\int_0^t EV(y(s\wedge\theta), r(s\wedge\theta))\,\mathrm{d}s. \end{aligned} \tag{4.4}$$

Let $C(\xi, \tilde{h}, T, \tau, i_0) = V(\xi(0), i_0) + \tilde{h}T + \sup_{i\in S, -\tau\leqslant u\leqslant 0} EV(\xi(u), i)\tilde{h}\tau$ and use the Gronwall inequality, we obtain

$$E[V(y(T\wedge\theta), r(T\wedge\theta))] \leqslant C(\xi, \tilde{h}, T, \tau, i_0)\mathrm{e}^{\tilde{h}(1+q)T}. \tag{4.5}$$

Let

$$v_R = \inf\{V(x, i) : |x| \geqslant R, i \in S\}.$$

By condition (i), $v_R \to \infty$ as $R \to \infty$. Noting that $|y(\theta)| = R$ whenever $\theta < T$, we derive from (4.5) that

$$C(\xi, \tilde{h}, T, \tau, i_0)\mathrm{e}^{\tilde{h}(1+q)T} \geqslant E[V(y(\theta), r(\theta))I_{\{\theta<T\}}] \geqslant v_R P(\theta < T).$$

That is

$$P(\theta < T) \leqslant \frac{C(\xi, \tilde{h}, T, \tau, i_0)\mathrm{e}^{\tilde{h}(1+q)T}}{v_R}. \tag{4.6}$$

Letting $R \to \infty$, we have

$$\lim_{t\to\infty} P(\theta < T) = 0,$$

this implies that the solution will not explode on $[0, T]$.

Step 2: For sufficiently large R define the stopping time

$$\rho = \inf\{t \in [0, T] : V(X(t), r(t)) \geqslant R\}.$$

Using (1.5) and applying the generalised Itô's formula to $V(X(t), r(t))$ yields

$$\begin{aligned}
&E[V(X(\rho \wedge t), r(\rho \wedge t))]\\
&\quad = V(\xi(0), i_0) + E\int_0^{\rho\wedge t}\Bigg[V_x(X(s), r(s))f(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))\\
&\qquad + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))V_{xx}(X(s), r(s))g(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))]\\
&\qquad + \sum_{j=1}^{N}\gamma_{r(s)j}V(X(s), j)\Bigg]\,\mathrm{d}s\\
&\quad \leqslant V(\xi(0), i_0) + E\int_0^{\rho\wedge t}\mathscr{L}V(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t}|V_x(X(s), r(s)) - V_x(\bar{X}_1(s), \bar{r}(s))||f(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))|\,\mathrm{d}s\\
&\qquad + \frac{1}{2}E\int_0^{\rho\wedge t}\mathrm{trace}[g^{\mathrm{T}}(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))[V_{xx}(X(s), r(s))\\
&\qquad - V_{xx}(\bar{X}_1(s), \bar{r}(s))]g(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))]\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t}\sum_{j=1}^{N}[\gamma_{r(s)j}V(X(s), j) - \gamma_{\bar{r}(s)j}V(\bar{X}_1(s), j)]\,\mathrm{d}s.
\end{aligned}$$

Rearranging the terms on the right-hand side by plus-and-minus technique and using condition (i) and (iii) we obtain that

$$
\begin{aligned}
&E[V(X(\rho\wedge t), r(\rho\wedge t))]\\
&\quad\leqslant V(\xi(0), i_0) + \tilde{h}T + \tilde{h}E\int_0^{\rho\wedge t} V(X(s), r(s))\,\mathrm{d}s + \tilde{h}E\int_0^{\rho\wedge t} V(X(s-\tau), r(s))\,\mathrm{d}s\\
&\qquad + \tilde{h}E\int_0^{\rho\wedge t} |V(\bar{X}_1(s), \bar{r}(s)) - V(X(s), \bar{r}(s))|\,\mathrm{d}s\\
&\qquad + \tilde{h}E\int_0^{\rho\wedge t} |V(X(s), \bar{r}(s)) - V(X(s), r(s))|\,\mathrm{d}s\\
&\qquad + \tilde{h}E\int_0^{\rho\wedge t} |V(\bar{X}_2(s), \bar{r}(s)) - V(X(s-\tau), \bar{r}(s))|\,\mathrm{d}s\\
&\qquad + \tilde{h}E\int_0^{\rho\wedge t} |V(X(s-\tau), \bar{r}(s)) - V(X(s-\tau), r(s))|\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t} |V_x(X(s), r(s)) - V_x(\bar{X}_1(s), r(s))||f(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))|\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t} |V_x(\bar{X}_1(s), r(s)) - V_x(\bar{X}_1(s), \bar{r}(s))||f(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))|\,\mathrm{d}s\\
&\qquad + \frac{1}{2}E\int_0^{\rho\wedge t} |V_{xx}(X(s), r(s)) - V_{xx}(\bar{X}_1(s), r(s))||g(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))|^2\,\mathrm{d}s\\
&\qquad + \frac{1}{2}E\int_0^{\rho\wedge t} |V_{xx}(\bar{X}_1(s), r(s)) - V_{xx}(\bar{X}_1(s), \bar{r}(s))||g(\bar{X}_1(s), \bar{X}_2(s), \bar{r}(s))|^2\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t} \sum_{j=1}^{N} |\gamma_{r(s)j}||V(X(s), j) - V(\bar{X}_1(s), j)|\,\mathrm{d}s\\
&\qquad + E\int_0^{\rho\wedge t} \sum_{j=1}^{N} |\gamma_{r(s)j} - \gamma_{\bar{r}(s)j}||V(\bar{X}_1(s), j)|\,\mathrm{d}s.
\end{aligned}
\tag{4.7}
$$

By condition (iv) we have

$$
\begin{aligned}
&E\int_0^{\rho\wedge t} [V(\bar{X}_1(s), \bar{r}(s)) - V(X(s), \bar{r}(s))]\,\mathrm{d}s\\
&\quad\leqslant E\int_0^{\rho\wedge t} K_R|\bar{X}_1(s) - X(s)|\,\mathrm{d}s \leqslant K_R\int_0^T E|V(\bar{X}_1(s), \bar{r}(s)) - V(X(s), \bar{r}(s))|\,\mathrm{d}s\\
&\quad\leqslant K_R\int_0^T (E|\bar{X}_1(s) - X(s)|^2)^{1/2}\,\mathrm{d}s.
\end{aligned}
$$

Using (2.5) we can show, in the same way as (2.6) was proved, that

$$E\int_0^{\rho\wedge t}[V(X(s),\bar{r}(s))-V(X(s),r(s))]\,\mathrm{d}s\leqslant 2V_R\gamma T(\Delta+\mathrm{o}(\Delta)),$$

and

$$E\int_0^{\rho\wedge t}[V(X(s-\tau),\bar{r}(s))-V(X(s-\tau),r(s))]\,\mathrm{d}s\leqslant 2V_R\gamma T(\Delta+\mathrm{o}(\Delta)),$$

where $\gamma=\max_{1\leqslant i\leqslant N}(-\gamma_{ii})$, $V_R=\sup\{V(x,i):|x|\leqslant R,\ i\in S\}$. We can similarly estimate the other terms on the right-hand side of (4.7) to get that

$$\begin{aligned}E[V(X(\rho\wedge t),r(\rho\wedge t))]&\\ \leqslant C(\xi,\tilde{h},T,\tau,i_0)&+\tilde{h}(1+q)E\int_0^{\rho\wedge t}V(X(s),r(s))\,\mathrm{d}s\\ &+C_1(R)\int_0^T(E|\bar{X}_1(\rho\wedge s)-X(\rho\wedge s)|^2)^{1/2}\,\mathrm{d}s\\ &+C_1(R)\int_0^T(E|\bar{X}_2(\rho\wedge s)-X(\rho\wedge(s-\tau))|^2)^{1/2}\,\mathrm{d}s+C_1(R)(\Delta+\mathrm{o}(\Delta)),\end{aligned}\tag{4.8}$$

where $C_1(R)$ and the following $C_2(R)$, $C_3(R)$, … are all constants dependent on R but independent of Δ. Moreover, in the same way as (2.10) was proved, we can show that

$$E|\bar{X}_1(\rho\wedge s)-X(\rho\wedge s)|^2\leqslant C_2(R)\Delta\quad\forall s\in[0,T],$$
$$E|\bar{X}_2(\rho\wedge s)-X(\rho\wedge(s-\tau))|^2\leqslant C_2(R)\Delta\quad\forall s\in[0,T].$$

Substituting these into (4.8) yields that

$$\begin{aligned}E[V(X(\rho\wedge t),r(\rho\wedge t))]\leqslant C(\xi,\tilde{h},T,\tau,i_0)&+C_3(R)(\Delta^{1/2}+\mathrm{o}(\Delta))\\ &+\tilde{h}(1+q)\int_0^t EV(X(\rho\wedge s),r(\rho\wedge s))\,\mathrm{d}s.\end{aligned}$$

By the Gronwall inequality,

$$E[V(X(\rho\wedge T),r(\rho\wedge T))]\leqslant \mathrm{e}^{\tilde{h}(1+q)T}[C(\xi,\tilde{h},T,\tau,i_0)+C_3(R)(\Delta^{1/2}+\mathrm{o}(\Delta))].\tag{4.9}$$

In the way as (4.6) was obtained, we can then show that

$$P(\rho<T)\leqslant\frac{\mathrm{e}^{\tilde{h}(1+q)T}}{v_R}[C(\xi,\tilde{h},T,\tau,i_0)+C_3(R)(\Delta^{1/2}+\mathrm{o}(\Delta))].\tag{4.10}$$

Step 3: Let $\beta=\rho\wedge\theta$. In the same way as Theorem 2.1 was proved we can show that

$$E\left[\sup_{0\leqslant t\leqslant\beta\wedge T}|X(t)-y(t)|^2\right]\leqslant C_4(R)(\Delta+\mathrm{o}(\Delta)).\tag{4.11}$$

Now, let $\varepsilon, \delta \in (0,1)$ be arbitrarily small. Set

$$\bar{\Omega} = \left\{ \omega : \sup_{0 \leqslant t \leqslant T} |X(t) - y(t)|^2 \geqslant \delta \right\}.$$

Using (4.11), we compute

$$\delta P(\bar{\Omega} \cap \{\beta \geqslant T\}) = \delta E\left[I_{\{\beta \geqslant T\}} I_{\bar{\Omega}} \right] \leqslant E\left[I_{\{\beta \geqslant T\}} \sup_{0 \leqslant t \leqslant \beta \wedge T} |X(t) - y(t)|^2 \right]$$
$$\leqslant E\left[\sup_{0 \leqslant t \leqslant \beta \wedge T} |X(t) - y(t)|^2 \right] \leqslant C_4(R)(\Delta + \mathrm{o}(\Delta)).$$

This, together with (4.6) and (4.10), yields that

$$P(\bar{\Omega}) \leqslant P(\bar{\Omega} \cap \{\beta \geqslant T\}) + P(\beta < T) \leqslant P(\bar{\Omega} \cap \{\beta \geqslant T\}) + P(\theta < T) + P(\rho < T)$$
$$\leqslant \frac{C_4(R)}{\delta}(\Delta + \mathrm{o}(\Delta)) + \frac{\mathrm{e}^{\tilde{h}(1+q)T}}{v_R} C_3(R)(\Delta^{1/2} + \mathrm{o}(\Delta)) + \frac{2C(\xi, \tilde{h}, T, \tau, i_0)\mathrm{e}^{\tilde{h}(1+q)T}}{v_R}.$$

Recalling that $v_R \to \infty$ as $R \to \infty$, we can choose R sufficiently large for

$$\frac{2C(\xi, \tilde{h}, T, \tau, i_0)\mathrm{e}^{\tilde{h}(1+q)T}}{v_R} < \frac{\varepsilon}{2},$$

and then choose Δ sufficiently small for

$$\frac{C_4(R)}{\delta}(\Delta + \mathrm{o}(\Delta)) + \frac{\mathrm{e}^{hT}}{v_R} C_3(R)(\Delta^{1/2} + \mathrm{o}(\Delta)) < \frac{\varepsilon}{2}$$

to obtain

$$P(\bar{\Omega}) = P\left(\sup_{0 \leqslant t \leqslant T} |X(t) - y(t)|^2 \geqslant \delta \right) < \varepsilon.$$

This proves the assertion (4.3). □

General speaking, the solution, $y(t), t \in [0, T]$, with initial data $\xi(t) \in C([-\tau, 0], G)$, $i_0 \in S$, to Eq. (1.1) is a member of an open set $G \subset \Re^m$, G is called the invariant set of Eq. (1.1). We note that G is often the positive cone $\Re^m_+$ or the whole Euclidean space $\Re^m$. In fact, from the lemma below we know that the solution to the stochastic delay Lotka–Volterra system (4.1) is a member of $\Re^m_+$.

Lemma 4.1. *Assume that there are positive numbers $c_1(i), \ldots, c_m(i)$, $i \in S$ and θ such that*

$$\lambda_{\max}\left(\frac{1}{2}[C(i)A(i) + A^{\mathrm{T}}(i)C(i) + \sigma^{\mathrm{T}}(i)C(i)\bar{Y}\sigma(i)] + \frac{1}{4\theta} C(i)B(i)B^{\mathrm{T}}(i)C(i) + \theta I \right) \leqslant 0, \quad (4.12)$$

where $C(i) = \mathrm{diag}(c_1(i), \ldots, c_m(i))$ and $\bar{Y} = \mathrm{diag}(\bar{y}_1, \ldots, \bar{y}_m)$, and I is the $m \times m$ identity matrix. Then for any given initial data $y(t) = \xi(t) \in C([-\tau, 0]; \Re^m_+)$, $t \in [-\tau, 0]$, $r(0) = i_0 \in S$, the solution $y(t)$ to Eq. (4.1) *on $0 \leqslant t \leqslant T$ will remain in $\Re^m_+$ with probability* 1.

Before we prove the lemma, we shall provide an example of matrices $A(i)$, $B(i)$ and $\sigma(i)$ such that condition (4.12) is satisfied. Let

$$A(1)=\begin{pmatrix}-4 & 0\\ 0 & -5\end{pmatrix},\quad B(1)=\begin{pmatrix}0 & -1\\ -2 & 0\end{pmatrix},\quad \sigma(1)=\begin{pmatrix}\frac{1}{4} & 0\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix},$$

$$A(2)=\begin{pmatrix}-10 & 1\\ 2 & -8\end{pmatrix},\quad B(2)=\begin{pmatrix}1 & 0\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},\quad \sigma(2)=\begin{pmatrix}\frac{1}{10} & \frac{1}{10}\\ 0 & \frac{1}{10}\end{pmatrix}.$$

It is not difficult to verify that

$$C(1)=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix},\quad C(2)=\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{2}\end{pmatrix},\quad \bar{Y}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

is a solution to (4.12).

Proof of Lemma 4.1. Let k_0 be sufficiently large for

$$\frac{1}{k_0}<\min_{-\tau\leqslant t\leqslant 0}|\xi(t)|\leqslant\max_{-\tau\leqslant t\leqslant 0}|\xi(t)|<k_0.$$

For $k\geqslant k_0$, define the stopping time

$$\rho_k=\inf\{t\geqslant 0: y_n(t)\notin(1/k,k)\text{ for some } n=1,\ldots,m\}.$$

Also define V on $\Re^m_+\times S$ such that

$$V(x,i)=\sum_{n=1}^{m}c_n(i)\bar{y}_n h(x_n/\bar{y}_n),\quad (x,i)\in\Re^m_+\times S, \tag{4.13}$$

where $h(u)=u-1-\ln(u)$. By (4.2), the operator associated with Eq. (4.1) is

$$\begin{aligned}\mathscr{L}V(x,y,i)=&\frac{1}{2}(x-\bar{y})^{\mathrm{T}}[C(i)A(i)+A^{\mathrm{T}}(i)C(i)+\sigma^{\mathrm{T}}(i)C(i)\bar{Y}\sigma(i)](x-\bar{y})\\ &+(x-\bar{y})^{\mathrm{T}}C(i)B(i)(y-\bar{y})+\sum_{j=1}^{N}\gamma_{ij}V(x,j).\end{aligned}\tag{4.14}$$

Noting that

$$(x-\bar{y})^{\mathrm{T}}C(i)B(i)(y-\bar{y})\leqslant\frac{1}{4\theta}(x-\bar{y})^{\mathrm{T}}C(i)B(i)B^{\mathrm{T}}(i)C(i)(x-\bar{y})+\theta|y-\bar{y}|^2$$

and the condition (4.12), we have

$$\begin{aligned}
\mathscr{L}V(x,y,i) \leqslant (x-\bar{y})^{\mathrm{T}} \bigg[& \frac{1}{2}(C(i)A(i)+A^{\mathrm{T}}(i)C(i)+\sigma^{\mathrm{T}}(i)C(i)\bar{Y}\sigma(i)) \\
& + \frac{1}{4\theta}C(i)B(i)B^{\mathrm{T}}(i)C(i)+\theta I \bigg](x-\bar{y}) \\
& - \theta|x-\bar{y}|^2+\theta|y-\bar{y}|^2+\sum_{j=1}^{N}\gamma_{ij}V(x,j) \\
\leqslant & -\theta|x-\bar{y}|^2+\theta|y-\bar{y}|^2+\sum_{j=1}^{N}\gamma_{ij}V(x,j).
\end{aligned} \tag{4.15}$$

Let

$$\hat{q} = \max\left\{ \frac{c_n(i)}{c_n(j)} : n=1,\ldots,m,\ i,\ j\in S \right\}.$$

By the definition of V, for any $i,\ j \in S$, we have

$$\begin{aligned}
\hat{q}V(x,i) &= \sum_{n=1}^{m} \hat{q}c_n(i)[(x_n-\bar{y}_n\ln x_n)+(\bar{y}_n\ln\bar{y}_n-\bar{y}_n)] \\
&\geqslant \sum_{n=1}^{m} c_n(j)[(x_n-\bar{y}_n\ln x_n)+(\bar{y}_n\ln\bar{y}_n-\bar{y}_n)] = V(x,j).
\end{aligned}$$

Hence

$$\begin{aligned}
E\int_0^{\rho_k\wedge t}\mathscr{L}V(y(s),y(s-\tau),r(s))\,\mathrm{d}s \leqslant E\int_0^{\rho_k\wedge t} & [-\theta|y(s)-\bar{y}|^2+\theta|y(s-\tau)-\bar{y}|^2 \\
& + \gamma\hat{q}NV(y(s),r(s))]\,\mathrm{d}s.
\end{aligned} \tag{4.16}$$

Compute

$$\begin{aligned}
E\int_0^{\rho_k\wedge t}|y(s-\tau)-\bar{y}|^2\,\mathrm{d}s &= E\int_{-\tau}^{\rho_k\wedge t-\tau}|y(s)-\bar{y}|^2\,\mathrm{d}s \\
&\leqslant \int_{-\tau}^{0}|y(s)-\bar{y}|^2\,\mathrm{d}s + E\int_0^{\rho_k\wedge t}|y(s)-\bar{y}|^2\,\mathrm{d}s.
\end{aligned}$$

Substituting this into (4.16) gives

$$\begin{aligned}
E\int_0^{\rho_k\wedge t}\mathscr{L}V(y(s),y(s-\tau),r(s))\,\mathrm{d}s \leqslant \theta & \int_{-\tau}^{0}|y(s)-\bar{y}|^2\,\mathrm{d}s \\
& + \gamma\hat{q}N\int_0^t EV(y(\rho_k\wedge s),r(\rho_k\wedge s))\,\mathrm{d}s.
\end{aligned} \tag{4.17}$$

Using the generalised Itô formula

$$
\begin{aligned}
&EV(y(\rho_k \wedge t), r(\rho_k \wedge t)) \\
&= V(\xi(0), r(0)) + E\int_0^{\rho_k \wedge t} \mathscr{L}V(y(s), y(s-\tau), r(s))\,\mathrm{d}s \\
&\leqslant V(\xi(0), r(0)) + \theta \int_{-\tau}^0 |y(s) - \bar{y}|^2\,\mathrm{d}s + \gamma\hat{q}N \int_0^t EV(y(\rho_k \wedge s), r(\rho_k \wedge s))\,\mathrm{d}s. \qquad (4.18)
\end{aligned}
$$

By the Gronwall inequality,

$$EV(y(\rho_k \wedge T), r(\rho_k \wedge T)) \leqslant K := \mathrm{e}^{\gamma\hat{q}NT}\left[V(\xi(0), r(0)) + \theta\int_{-\tau}^0 |\xi(s) - \bar{y}|^2\,\mathrm{d}s\right].$$

Note that for every $\omega \in \{\rho_k \leqslant T\}$, there is some n such that $y_n(\rho_k, \omega)$ equals either k or $1/k$, hence

$$
\begin{aligned}
K &\geqslant E[I_{\{\rho_k \leqslant T\}} V(y(\rho_k, \omega), r(\rho_k, \omega))] \\
&\geqslant P(\rho_k \leqslant T) \min_{i\in S, 1\leqslant n\leqslant m} \left\{c_n(i)\bar{y}_n \left(\left[\frac{k}{\bar{y}_n} - 1 - \ln\left(\frac{k}{\bar{y}_n}\right)\right] \wedge \left[\frac{1}{k\bar{y}_n} - 1 - \ln k\bar{y}_n\right]\right)\right\}.
\end{aligned}
$$

Letting $k \to \infty$ gives

$$\lim_{k\to\infty} P(\rho_k \leqslant T) = 0,$$

this implies that $y(t) \in \Re^m_+$. The proof is therefore complete. $\square$

Observing from the proof of Theorem 4.1 that condition (i) and condition (iii) are used to derive (4.4) and (4.8), we rewrite Theorem 4.1 as the following theorem, i.e.

Theorem 4.2. *Let G be an open subset of $\Re^m$, and Let $y(t)$ denote the solution of* (1.1) *for $t \in [0, T]$ with initial data $\xi \in ([-\tau, 0]; G)$, $i_0 \in S$. Moreover, assume that $y(t)$ is a member of G. Let $D \subset G$ be any compact set. Suppose the following conditions are satisfied*:

(i) *there exists a positive constant L_D such that $x, y \in D$, $i \in S$*

$$|f(x, y, i) - f(\bar{x}, \bar{y}, i)| \vee |g(x, y, i) - g(\bar{x}, \bar{y}, i)| \leqslant L_D(|x - \bar{x}| + |y - \bar{y}|)$$

(ii) $\lim_{|x|\to\infty} V(x, i) = \infty$ *for any $i \in S$ and $\{x \in G : V(x, i) \leqslant R\}$ is compact for any $R > 0$, $i \in S$*;

(iii) *Let α be a bounded stopping time, if $V(y(s), r(s))$ and $\mathscr{L}V(y(s), y(s-\tau), r(s))$ are bounded in* $[0, \alpha \wedge T]$, *there exists a positive constant $\hat{h}$ such that*

$$E\int_0^{\alpha\wedge T} \mathscr{L}V(y(s), y(s-\tau), r(s))\,\mathrm{d}s \leqslant \hat{h} + \hat{h}E\int_0^{\alpha\wedge T} V(y(s), r(s))\,\mathrm{d}s;$$

(iv) *there exists a positive constant K_D such that for all $i \in S$ and those $x, y \in D$*

$$|V(x, i) - V(y, i)| \vee |V_x(x, i) - V_x(y, i)| \vee |V_{xx}(x, i) - V_{xx}(y, i)| \leqslant K_D|x - y|.$$

Then the EM approximate solution converges to the exact solution of the SDDEwMS (1.1) *in the sense that*

$$\lim_{\Delta\to 0}\left(\sup_{0\leqslant t\leqslant T}|X(t)-y(t)|^2\right)=0 \quad \textit{in probability.} \tag{4.19}$$

Let us return to the stochastic delay Lotka–Volterra Eq. (4.1) again. Let $y(t)$ denote the solution to Eq. (4.1). By Lemma 4.1, $y(t)$ is a member of $\Re^m_+$. Let $D \subset \Re^m_+$ be any compact set and define $V(x,i)$ define (4.13). It is easy to see the coefficients of Eq. (4.1) satisfy condition (i) of Theorem 4.2 and $V(x,i)$ satisfies condition (ii) of Theorem 4.2. Under the condition of Lemma 4.1, by (4.17) we know that condition (iii) of Theorem 4.2 holds. Since the partial derivatives of $V(x,i)$ are bounded in compact set D, the condition (iv) of Theorem 4.2 holds as well. Therefore, under the condition of Theorem 4.2 the EM solution of Eq. (4.1) will converge to its exact solution in probability.

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